Inferring the optimal exercise boundary of an American option

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American put/call options are financial instruments that provide the holder the right, but not the obligation, to sell/buy a given underlying financial asset (e.g., a stock) for a constant specific (strike) price $S > 0$ at any time before an expiration date $T > 0$.

To find the best exercise strategy for an American option, it is common to assume that the price of the underlying asset is modeled by a Stochastic Differential Equation (SDE):

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t, \quad t \in [0, T],$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion, and μ and σ are the functions representing the drift and volatility of the asset $(X_t)_{t\in[0,T]}$.

In such a case, the best time for exercising the corresponding American put option often coincides with the first time X_t steps below the graph of a certain function $b : [0, T] \to \mathbb{R}$, called the Optimal Stopping Boundary (OSB). See Peskir (2005) for a first mathematical treatment of the American option problem with a geometric Brownian motion, that is, $\mu(t, x) = rx$ and $\sigma(t, x) = cx$, for $r, c, x > 0$. Figure 1 illustrates the OSB and the process $(X_t)_{t\in[0,T]}$.

Under some conditions, b can be defined as the unique solution of the (free-boundary) integral equation:

$$b(t) = S - \mathbb{E}_{t,b(t)}\left[e^{-\lambda(T-t)}(S - X_T)^+\right] + \int_t^T K_\lambda(t, b(t), u, b(t + u))\,\mathrm{d}u,$$

for the kernel

$$K_\lambda(t, x, u, y) = e^{-\lambda u}\mathbb{E}_{t,x}\left[(\lambda(S - X_{t+u}) + \mu(t + u, X_{t+u}))\, I(X_{t+u} \leq y)\right],$$

where $\mathbb{E}_{t,x}$ is the mean operator under the condition $X_t = x$. This equation can be solved numerically in practice.

This thesis proposal aims to make inference of the OSB associated with American options under different parametrizations of the drift and the volatility of the underlying SDE process models, as well as carrying out real data studies to test the different models' performances in real-life scenarios. The goal is to implement algorithms to estimate μ and σ based on the evolution of X_t during an initial time interval $[0, T']$, for $T' \in (0, T)$. Those approximations would then be used to estimate $b(t)$ in the remaining interval $[T', T]$, by solving the free-boundary equation numerically. The OSB's estimation error could

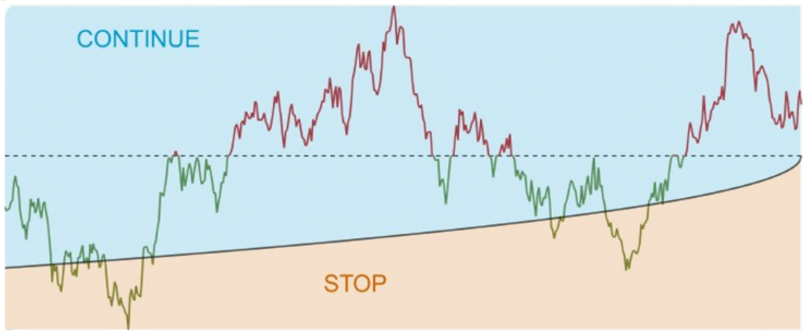

Figure 1: The OSB (continuous black line) splits the time-price space into two complementary regions: the stopping set, in light orange, which lies above the OSB and includes all the pairs (t, x) where it is optimal to exercise the option; and the continuation set, in light blue. The asset's price is colored in green whenever it is profitable (but not necessarily optimal) to exercise the option, that is below the strike price (dashed line) and in red otherwise. Optimal stopping happens when the stock price first enters the stopping set.

be controlled by relying on the Delta method or bootstrap resampling techniques. The management of the error could be used to propose alternative stopping strategies. For example, one could find $b_\alpha : [0, T] \to \mathbb{R}$ such that

$$\mathbb{P}[b(t) \leq b_\alpha(t)] \approx 1 - \alpha.$$

Besides illustrating the whole methodology in simulated scenarios, we will collect financial real-data to test the different models and estimated stopping strategies. Potential secondary goals of the thesis proposal might include obtaining some theoretical properties (see Peskir & Shiryaev (2006)) of the OSB and a formal validation of the free-boundary equation. The interested student will build upon previous work (Azze et al., 2024; D'Auria et al., 2020) and have access to software already developed (https://github.com/aguazz/AmOpBB, https://github.com/aguazz/OSP_OUB).

References

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