Optimal stopping of an Ornstein–Uhlenbeck bridge in mathematical finance

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American put/call options are financial instruments that provide the holder the right, but not the obligation, to sell/buy a given underlying financial asset (e.g., a stock) for a constant specific (strike) price $S > 0$ at any time before an expiration date $T > 0$.

To find the best exercise strategy for an American option, it is common to assume that the price of the underlying asset is modeled by a Stochastic Differential Equation (SDE):

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}W_t, \quad t \in [0, T],$$

where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion, and μ and σ are the functions representing the drift and volatility of the asset $(X_t)_{t\in[0,T]}$.

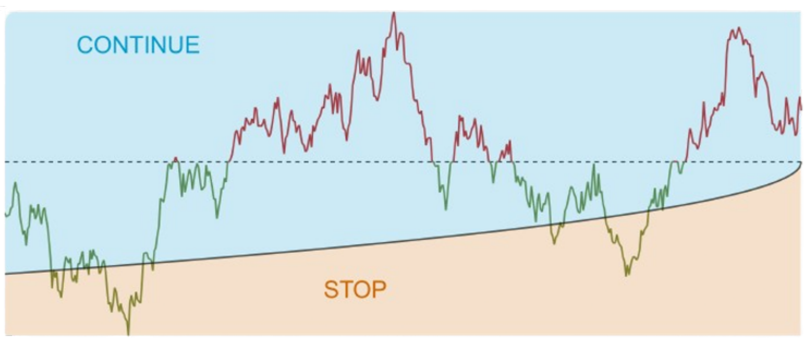

Figure 1: The OSB (continuous black line) splits the time-price space into two complementary regions: the stopping set, in light orange, which lies above the OSB and includes all the pairs (t, x) where it is optimal to exercise the option; and the continuation set, in light blue. The asset's price is colored in green whenever it is profitable (but not necessarily optimal) to exercise the option, that is below the strike price (dashed line) and in red otherwise. Optimal stopping happens when the stock price first enters the stopping set.

The best time for exercising the corresponding American put option often coincides with the first time X_t steps below/above the graph of certain function $b : [0, T] \to \mathbb{R}$, called the Optimal Stopping Boundary (OSB). See Peskir (2005) for a first mathematical treatment of the American option problem with a geometric Brownian motion, that is, $\mu(t,x) = rx$ and $\sigma(t,x) = cx$, for $r, c, x > 0$. Figure 1 illustrates the OSB and the process $(X_t)_{t\in[0,T]}$. Under some conditions, b can be defined as the unique solution of a (free-boundary) integral equation, which can be solved numerically in practice.

This thesis proposal aims to explore the profit associated to exercising American options optimally when the underlying asset behaves like an Ornstein–Uhlenbeck (OU) process and an Ornstein–Uhlenbeck Bridge (OUB). An OU process, associated to $\mu(t,x) = -\theta x$ and $\sigma(t,x) \equiv \nu$, for $\theta, \nu > 0$, is often times the go-to model for pair trading and volatility indexes. The OUB, which results after forcing an OU process to hit a deterministic value $z \in \mathbb{R}$ at a terminal time, is given by taking $\mu(t,x) = \theta \frac{z - \cosh(\theta(1-t))x}{\sinh(\theta(1-t))}$ and $\sigma(t,x) = \nu$, for $\theta, \nu > 0$. This latter process may prove itself useful for modeling the believes of an investor with insight information about the asset's price at the maturity date of the American option, which may find applications when trading perishable assets or within the so-called stock pinning effect (see, e.g., Avellaneda & Lipkin (2003), Ni et al. (2005), Jeannin et al. (2008), Avellaneda et al. (2012), Ni et al. (2021)).

While the theoretical treatment of the optimal exercise of American options under OU processes and OUBs can be found in Azze et al. (2024a) and Azze et al. (2024b), respectively, this thesis focuses more on the applicability and numerical aspects. It involves:

- Finding, collecting, and processing data of stock prices that, either directly or after transformations, fit the OU and OUB models.
- Computing the OSB associated to the optimal exercise of American options and calculating the profits made by following such optimal strategies.
- Assessing the model-robustness of the optimal strategies/profits.
- Comparing the profitability of assuming an OU/OUB model versus classical models in the literature.

The interested student must be proficient in data collecting, processing, and coding skills. Some statistical and mathematical knowledge would also be required. The thesis builds upon previous work (D'Auria et al. (2020), Azze et al. (2024a), Azze et al. (2024b)) and access to software already developed is available (https://github.com/aguazz/AmOpBB, https://github.com/aguazz/OSP_OUB, https://github.com/aguazz/AmOpTDOU).

References

Avellaneda, M., Kasyan, G., & Lipkin, M. D. (2012). Mathematical models for stock pinning near option expiration dates. *Communications on Pure and Applied Mathematics*, *65*(7), 949–974. https://doi.org/10.1002/cpa.21404

Avellaneda, M., & Lipkin, M. D. (2003). A market-induced mechanism for stock pinning. *Quantitative Finance*, *3*(6), 417–425. https://doi.org/10.1088/1469-7688/3/6/301

Azze, A., D'Auria, B., & García-Portugués, E. (2024a). Optimal exercise of American options under time-dependent Ornstein–Uhlenbeck processes. *Stochastics*, *96(1)*, 921–946,. https://doi.org/10.1080/17442508.2024.2325402

Azze, A., D'Auria, B., & García-Portugués, E. (2024b). Optimal stopping of an Ornstein–Uhlenbeck bridge. *Stochastic Processes and Their Applications*, *172*, 104342. https://doi.org/10.1016/j.spa.2024.104342

D'Auria, B., García-Portugués, E., & Guada, A. (2020). Discounted optimal stopping of a Brownian bridge, with application to American options under pinning. *Mathematics*, *8*(7), 1159. https://doi.org/10.3390/math8071159

Jeannin, M., Iori, G., & Samuel, D. (2008). Modeling stock pinning. *Quant. Finance*, *8*(8), 823–831. https://doi.org/10.1080/14697680701881763

Ni, S. X., Pearson, N. D., & Poteshman, A. M. (2005). Stock price clustering on option expiration dates. *J. Financ. Econ.*, *78*(1), 49–87. https://doi.org/10.1016/j.jfineco.2004.08.005

Ni, S. X., Pearson, N. D., Poteshman, A. M., & White, J. (2021). Does option trading have a pervasive impact on underlying stock prices? *Rev. Financ. Stud.*, *34*(4), 1952–1986. https://doi.org/10.1093/rfs/hhaa082

Peskir, G. (2005). On the American option problem. *Mathematical Finance*, *15*(1), 169–181. https://doi.org/10.1111/j.0960-1627.2005.00214.x